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TECHNICAL NOTE 3274

SOME LINEAR DYNAMICS OF TWO-SPOOL TURBOJET ENGINES

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Cleveland, Ohio

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SUMMARY

General equations for the linear responses of inner- and outer-spool speed to change in turbine-inlet temperature and exhaust-nozzle area are derived and evaluated from hypothetical two-spool-engine characteristics at design speed. The resultant equations of response are corroborated with experimental data. At design speed the response of inner-spool speed to turbine-inlet temperature approximated a first-order lag specified by the time constant of the inner spool. The outer-spool speed response to turbine-inlet temperature was found experimentally to be identical with the response of inner-spool speed, in that it approximated a first-order lag specified by the inner-spool time constant. The analytically derived response of outer-spool speed to turbine-inlet temperature was a lead second-order lag that could result in the experimentally determined response by cancellation of terms in the response equation. With respect to changes in exhaust-nozzle area, the design-speed response of the outer-spool speed approximated a first-order lag specified by the outer-spool time constant, whereas the inner-spool speed response approximated a second-order lag, but of negligible amplitude.

INTRODUCTION

Knowledge of the dynamic characteristics of a turbojet engine is a prerequisite to the design of a control system. Linear dynamics are useful in stability and control parameter considerations, and nonlinear dynamics are required for determination of surge limits and maximum acceleration potential.

Considerable effort has already been expended on the determination of the linear dynamics of single-spool turbojet engines; these engines are generally accepted as first-order lag systems characterizied by time constants (response of speed to fuel flow or exhaust-nozzle area). In the two-spool engine each spool acts as an energy storage element, and, hence, the two-spool engine is inherently a second-order system. Published information relative to the dynamics of the two-spool engine has thus far been scarce. In order to facilitate consideration of control systems for two-spool engines, a brief analysis of the linear response characteristics of this type of engine is presented in this report.

The analysis presented herein is concerned with the linear responses of the two spools to changes in turbine-inlet temperature at constant exhaust-nozzle area and to changes in exhaust-nozzle area at constant turbine-inlet temperature.

General equations of response are developed from linearization of functional relations. The general equations are then evaluated at design speed by means of representative engine thermodynamic relations. 4031

Corroboration of the results of the analysis is indicated by the use of experimental data.

ANALYSIS

Development of General Equations of Speed Response

The response of each spool to changes in either turbine-inlet temperature or exhaust-nozzle area may be obtained from linearization of the functional engine relations. (Terminology is in accordance with appendix A and fig. 1.)

The torque of each spool is assumed to be a function of four variables:

$$Q_o = Q_o(N_o, N_i, T_4, A) \tag{1a}$$

$$Q_i = Q_i(N_o, N_i, T_4, A) \tag{1b}$$

The functional relations of equations (1a) and (1b) can be expanded as follows:

$$\Delta Q_o = I_o \,\Delta\left(\frac{dN_o}{dt}\right) = I_o p \,\Delta N_o = \frac{\partial Q_o}{\partial N_o}\,\Delta N_o + \frac{\partial Q_o}{\partial N_i}\,\Delta N_i + \frac{\partial Q_o}{\partial T_4}\,\Delta T_4 + \frac{\partial Q_o}{\partial A}\,\Delta A \tag{2a}$$

$$\Delta Q_i = I_i \,\Delta\left(\frac{dN_i}{dt}\right) = I_i p \,\Delta N_i = \frac{\partial Q_i}{\partial N_o}\,\Delta N_o + \frac{\partial Q_i}{\partial N_i}\,\Delta N_i + \frac{\partial Q_i}{\partial T_4}\,\Delta T_4 + \frac{\partial Q_i}{\partial A}\,\Delta A \tag{2b}$$

Response to turbine-inlet temperature at constant exhaust-nozzle area. - Solution of equations (2a) and (2b) for $\Delta N_o/\Delta T_4$ and $\Delta N_i/\Delta T_4$ at $\Delta A = 0$ gives

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = \frac{\frac{\partial Q_o}{\partial T_4}\left(I_i p - \frac{\partial Q_i}{\partial N_i}\right) + \frac{\partial Q_i}{\partial T_4}\frac{\partial Q_o}{\partial N_i}}{\left(I_i p - \frac{\partial Q_i}{\partial N_i}\right)\left(I_o p - \frac{\partial Q_o}{\partial N_o}\right) - \frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}} \tag{3a}$$

$$\left.\frac{\Delta N_i}{\Delta T_4}\right|_A = \frac{\frac{\partial Q_i}{\partial T_4}\left(I_o p - \frac{\partial Q_o}{\partial N_o}\right) + \frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial T_4}}{\left(I_i p - \frac{\partial Q_i}{\partial N_i}\right)\left(I_o p - \frac{\partial Q_o}{\partial N_o}\right) - \frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}} \tag{3b}$$

Equations (3a) and (3b) can be rearranged as follows:

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\frac{\partial Q_o}{\partial T_4}}{\frac{\partial Q_o}{\partial N_o}}\;\frac{\left(-\frac{I_i}{\frac{\partial Q_i}{\partial N_i}}p + 1\right) - \frac{\frac{\partial Q_i}{\partial T_4}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial T_4}}}{\left(-\frac{I_i}{\frac{\partial Q_i}{\partial N_i}}p + 1\right)\left(-\frac{I_o}{\frac{\partial Q_o}{\partial N_o}}p + 1\right) - \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}}} \tag{4a}$$

$$\left.\frac{\Delta N_i}{\Delta T_4}\right|_A = -\frac{\frac{\partial Q_i}{\partial T_4}}{\frac{\partial Q_i}{\partial N_i}}\;\frac{\left(-\frac{I_o}{\frac{\partial Q_o}{\partial N_o}}p + 1\right) - \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial T_4}}{\frac{\partial Q_o}{\partial N_o}\frac{\partial Q_i}{\partial T_4}}}{\left(-\frac{I_i}{\frac{\partial Q_i}{\partial N_i}}p + 1\right)\left(-\frac{I_o}{\frac{\partial Q_o}{\partial N_o}}p + 1\right) - \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}}} \tag{4b}$$

Let $\tau_o = -\dfrac{I_o}{\dfrac{\partial Q_o}{\partial N_o}}$ and $\tau_i = -\dfrac{I_i}{\dfrac{\partial Q_i}{\partial N_i}}$ so that equations (4a) and (4b) become

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}} \frac{\left(\tau_i p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial N_o}}} \tag{5a}$$

and

$$\left.\frac{\Delta N_i}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_i}{\partial T_4}}{\dfrac{\partial Q_i}{\partial N_i}} \frac{\left(\tau_o p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}\dfrac{\partial Q_i}{\partial T_4}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial N_o}}} \tag{5b}$$

Response to exhaust-nozzle area at constant turbine-inlet temperature. - If equations (2a) and (2b) are used to determine speed response to exhaust-nozzle area at constant turbine-inlet temperature ($\Delta T_4 = 0$), the following equations are obtained in a manner identical with the preceding development:

$$\left.\frac{\Delta N_o}{\Delta A}\right|_{T_4} = -\frac{\dfrac{\partial Q_o}{\partial A}}{\dfrac{\partial Q_o}{\partial N_o}} \frac{\left(\tau_i p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial A}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial A}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial N_o}}} \tag{6a}$$

$$\left.\frac{\Delta N_i}{\Delta A}\right|_{T_4} = -\frac{\frac{\partial Q_i}{\partial A}}{\frac{\partial Q_i}{\partial N_i}} \frac{\left(\tau_o p + 1\right) + \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial A}}{\frac{\partial Q_i}{\partial A}\frac{\partial Q_o}{\partial N_o}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right) - \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}}} \tag{6b}$$

or, rearranged

$$\left.\frac{\Delta N_i}{\Delta A}\right|_{T_4} = \frac{-\frac{\frac{\partial Q_i}{\partial A}}{\frac{\partial Q_i}{\partial N_i}}\left(\tau_o p + 1\right) + \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial A}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right) - \frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}}} \tag{6b}$$

Simplified Equations of Speed Response at Design Conditions

The general equations for the responses of outer- and inner-spool speed to changes in turbine-inlet temperature and exhaust-nozzle area are represented by equations (5) and (6), respectively. The forms of the equations are such that the responses appear to be much more complicated than the first-order lag responses of a single-spool engine. The possibility exists, however, that the general responses might be considerably simpler if the magnitudes of some of the partial derivatives are small enough to be neglected. In order to explore the possible simplification of responses, the partials of equations (5) and (6) must be evaluated. This evaluation of the partial derivatives is presented in appendix B for a hypothetical two-spool engine at design conditions. The method of evaluation is based on engine thermodynamic relations and is limited to design conditions in order to eliminate the need for component maps and to justify assumptions that minimize evaluation procedures.

As indicated in appendix B, the quantities $\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial N_i}\Big/\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial N_o}$ and $\dfrac{\partial Q_i}{\partial N_o}\dfrac{\partial Q_o}{\partial T_4}\Big/\dfrac{\partial Q_o}{\partial N_o}\dfrac{\partial Q_i}{\partial T_4}$ are negligible, and the quantities $\dfrac{\partial Q_i}{\partial A}\dfrac{\partial Q_o}{\partial N_i}\Big/\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial A}$ and $\dfrac{\partial Q_i}{\partial A}$ are equal to zero. Equations (5) and (6), therefore, simplify as follows:

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}}\,\frac{\left(\tau_i p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right)} \tag{7a}$$

$$\left.\frac{\Delta N_i}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_i}{\partial T_4}}{\dfrac{\partial Q_i}{\partial N_i}}\,\frac{1}{\tau_i p + 1} \tag{7b}$$

$$\left.\frac{\Delta N_o}{\Delta A}\right|_{T_4} = -\frac{\dfrac{\partial Q_o}{\partial A}}{\dfrac{\partial Q_o}{\partial N_o}}\,\frac{1}{\tau_o p + 1} \tag{8a}$$

$$\left.\frac{\Delta N_i}{\Delta A}\right|_{T_4} = \frac{0.043\,\dfrac{N_i}{A}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right)}$$

or in percentage change,

$$\left.\frac{\dfrac{\Delta N_i}{N_i}}{\dfrac{\Delta A}{A}}\right|_{T_4} = \frac{0.043}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right)} \tag{8b}$$

Under the assumptions made in the evaluation of the partial derivatives (appendix B), the response equations have now been simplified considerably. The response of inner-spool speed to turbine-inlet temperature (eq. (7b)) and the response of outer-spool speed to exhaust-nozzle area reduce to first-order lags. The response of inner-spool speed to exhaust-nozzle area (eq. (8b)) reduces to a second-order lag, but the percentage magnitude of this response is so small (0.043-percent speed change for a 1-percent area change) that the net response is probably negligible despite the dynamics involved. The response of outer-spool speed to turbine-inlet temperature (eq. (7a)) simplifies only slightly and remains a lead second-order lag because the value of the grouped partial derivative term in the numerator is too large to be considered negligible.

EXPERIMENTAL CORROBORATION OF ANALYSIS

In order to obtain experimental corroboration of the responses indicated by equations (7) and (8), transient data from a two-spool engine were analyzed and the experimental transfer functions thereby obtained were compared with the derived transfer functions. Only the dynamic elements of the responses were compared because the equilibrium magnitude of the responses would be a function of a specific engine. Figure 2 shows the transients in inner- and outer-spool speeds for a change in fuel flow at constant exhaust-nozzle area. Figure 3 shows the transients in inner- and outer-spool speeds for a change in exhaust-nozzle area at constant fuel flow. Both sets of data were taken near maximum engine speed in order to be consistent with the limitations imposed upon the responses derived analytically.

Response to Fuel Flow

Although the data shown in figure 2 are for a change in fuel flow rather than in turbine-inlet temperature, the responses are expected to be very similar. These responses are similar because the transfer function of turbine-inlet temperature with respect to fuel flow is approximately unity in the top speed region. Transfer functions obtained from the data with respect to a change in fuel flow are therefore applicable for comparison with the transfer functions derived analytically for a change in turbine-inlet temperature.

From a harmonic analysis of the data shown in figure 2 the frequency responses of the inner- and outer-spool speeds to a change in fuel flow were obtained, and are shown in the amplitude-ratio and phase-shift plots of figure 4. The amplitude ratio at zero frequency was normalized to unity. As can be seen from figure 4, the frequency response of the inner-spool speed closely approximates a theoretical first-order lag and therefore corroborates equation (7b).

The frequency response of outer-spool speed, also shown in figure 4, appears to be identical with the inner-spool response; that is, the response of outer-spool speed appears to be a first-order lag specified by the time constant of the inner spool. The fact that the experimental response of outer-spool speed seems to be virtually the same as the response of inner-spool speed implies that the second-order system of equation (7a) must degenerate into a first-order system by cancellation of terms. It is assumed that this simplification occurs in the following manner:

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}}\;\frac{\left(\tau_i p + 1\right) - \dfrac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right)} \tag{7a}$$

This equation can be rearranged to

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}}\left(1 - \frac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}\right)\frac{\dfrac{\tau_i p}{1 - \dfrac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}} + 1}{\left(\tau_i p + 1\right)\left(\tau_o p + 1\right)}$$

However, in order to obtain a cancellation of the term $\left(\tau_o p + 1\right)$ in the denominator and thereby obtain a first-order lag of the form $G\,\dfrac{1}{1 + \tau_i p}$, the following relation must apply:

$$\frac{\tau_i}{1 - \dfrac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}} = \tau_o \tag{9}$$

(From appendix B, $\tau_i = 1.6\,\tau_o$.) Therefore,

$$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A = -\frac{\dfrac{\partial Q_o}{\partial T_4}}{\dfrac{\partial Q_o}{\partial N_o}}\left(1 - \frac{\dfrac{\partial Q_i}{\partial T_4}\dfrac{\partial Q_o}{\partial N_i}}{\dfrac{\partial Q_i}{\partial N_i}\dfrac{\partial Q_o}{\partial T_4}}\right)\left(\frac{1}{\tau_i p + 1}\right)$$

On the assumption that $\tau_i = \tau_o \left(1 - \frac{\partial Q_i}{\partial T_4} \frac{\partial Q_o}{\partial N_i} \middle/ \frac{\partial Q_i}{\partial N_i} \frac{\partial Q_o}{\partial T_4}\right)$ the lead second-order lag response of equation (7a) can therefore be simplified into the first-order lag response that was determined experimentally. Because the time constants are functions of engine speed and moments of inertia, and since these physical engine characteristics are not assigned to the hypothetical two-spool engine assumed in the analysis, the cancellation of terms based on the relation between τ_i and τ_o could not be predicted analytically. However, for actual engines, the analytical relation required between τ_i and τ_o for cancellation of terms in the derived equation (7a) has been found to correspond with measured values of these time constants.

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Response to Exhaust-Nozzle Area

Examination of the traces in figure 3 shows that although a change in outer-spool speed resulted from the change in exhaust-nozzle area, there was no perceptible change in inner-spool speed. Figure 3 therefore indicates that the dynamics of the inner spool, with respect to a change in exhaust-nozzle area at constant fuel flow, are negligible. The fact that essentially no dynamics were obtained for inner-spool response corroborates the small gain term determined analytically in equation (8b).

In attempting to determine the response of outer-spool speed to exhaust-nozzle area, it was found that the traces of figure 3 could not be used for harmonic analysis. The exhaust-nozzle area was actually indicated by measurement of the exhaust-nozzle position, and the position did not bear a linear relation to the exhaust-nozzle area. If the exhaust nozzle had been positioned rapidly enough to approach a step change in area, this nonlinearity could have been neglected, but, as shown in figure 3, the change in the exhaust-nozzle position was quite slow.

If the response of outer-spool speed to a change in exhaust-nozzle area is a first-order lag, as indicated by equation (8a), the speed trace remaining at the conclusion of the ramp change in exhaust-nozzle area should be exponential and should plot as a straight line on semilog paper. With zero time considered as the point at which the exhaust-nozzle area reached a constant value, the outer-spool speed trace was plotted on semilog paper as indicated in figure 5. It can be seen that the plot is essentially a straight line, and it can therefore be concluded that the response of outer-spool speed to a change in exhaust-nozzle area is a first-order lag.

It is to be noted that the time constant of the outer spool, as obtained from figure 5, is significantly smaller than the time constant of the inner spool, obtained from figure 4.

DISCUSSION OF RESULTS

Engine responses to changes in turbine-inlet temperature and exhaust-nozzle area are indicated by equations (5) and (6), respectively. Under the assumed conditions of linearity, the responses to simultaneous changes in both independent variables are obtained by the addition of the separate responses (principle of linear superposition).

Response to Turbine-Inlet Temperature

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The design speed response of inner-spool speed to turbine-inlet temperature at constant exhaust-nozzle area (eq. (7b)) approximated a first-order lag specified by the time constant of the inner spool.

It is a characteristic of two-spool engines that near design speed the thrust is essentially a function only of the inner-spool speed (independent of outer-spool speed and exhaust-nozzle area). Because thrust control may therefore be achieved by regulating the inner-spool speed and because the inner-spool speed response is first order, it appears that thrust control of a two-spool engine is a problem only of controlling a first-order system just as it is in a single-spool engine.

Analysis (eq. (7a)) shows the design speed response of an outer-spool speed to be a lead second-order lag. However, experimental evidence indicates that cancellation of terms results in reduction of this response to a first-order lag, again specified by the time constant of the inner spool. This cancellation of terms is based upon a required relation existing between inner- and outer-spool time constants that has been verified on actual two-spool engines, but could not be predicted analytically without a more specific description of the hypothetical two-spool engine assumed for analysis. It has been noted that the time constant of the outer spool, obtained from figure 5, is significantly smaller than the time constant of the inner spool, obtained from figure 4. This relation between inner- and outer-spool time constants, which was obtained experimentally, tends to verify the analytical relation of equation (9) and therefore indicates the validity of cancellation of terms.

The fact that the two spools respond to a change in turbine-inlet temperature in accordance with the inner-spool time constant implies that the inner spool represents the driving impetus of the two-spool combination. This reasoning can be borne out by the fact that the effect of outer-spool speed changes on inner-spool torque is very small compared with the effect of inner-spool speed changes on outer-spool torque. That is,

$$\frac{\dfrac{\partial Q_i}{\partial N_o}}{\dfrac{\partial Q_o}{\partial N_i}} = \frac{\dfrac{w}{N_o N_i} C\left(\dfrac{H_2}{j - \frac{1}{2} - \frac{D}{2}} - \dfrac{H_3}{k}\right)}{\dfrac{w}{N_o N_i} \dfrac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}} = 0.048$$

(from appendix B).

The relation between inner- and outer-spool responses also implies that the steady-state speed correspondence between the two spools will be closely maintained during transients caused by changes in turbine-inlet temperature.

Response to Exhaust-Nozzle Area

The outer-spool response (at design speed) to a change in exhaust-nozzle area at constant turbine-inlet temperature, indicated in equation (8a), is a simple first-order lag specified by the outer-spool time constant. The response of inner-spool speed is a second-order lag but of extremely small magnitude, so that for a change in exhaust-nozzle area the dynamics of the outer spool determine the engine response. From the control standpoint, these response characteristics imply that there will be little effect on the inner-spool speed (and, hence, engine thrust) during initiation of afterburner operation.

Restrictions of Analysis

Evaluation of the general equations of response is based on the assumption of choking at both turbines and at the exhaust nozzle. This assumption is reasonable only for engine operation near top speed, therefore, evaluations and conclusions would be valid only near top-speed conditions. However, it is a characteristic of two-spool engines that virtually the entire useable thrust output is attained within the top 10 percent of the speed range (for subsonic operation).

The design details of the hypothetical two-spool engine assumed herein do not materially affect the results inasmuch as similar results have been obtained for a diversity of assumed engines. The engine used for experimental corroboration of the analysis was, in fact, considerably different from the hypothetical engine used for analysis. It is noted that many assumed values for engine characteristics cancel out in the evaluation of the partial derivatives. The assumed compressor characteristic specifying corrected air flow as a function only of corrected engine speed has been found valid for many compressors at, or near, the top-speed condition for fixed inlet conditions.

SUMMARY OF RESULTS

General equations for the responses of inner- and outer-spool speed to changes in turbine-inlet temperature and exhaust-nozzle area have been derived. These general equations have been evaluated on the basis of hypothetical two-spool engine characteristics at design conditions; and the design-speed response characteristics obtained and corroborated with experimental data are summarized as follows: 4031

1. The response of inner-spool speed to turbine-inlet temperature was found to approximate a first-order lag specified by the inner-spool time constant.

2. The response of outer-spool speed to turbine-inlet temperature was found analytically to approximate a lead second-order lag. However, experimental evidence indicates that cancellation of terms causes the response of the outer-spool speed to the turbine-inlet temperature to appear as a first-order lag, again specified by the time constant of the inner spool.

3. The response of outer-spool speed to exhaust-nozzle area approximated a first-order lag specified by the outer-spool time constant.

4. The response of inner-spool speed to exhaust-nozzle area approximated a second-order lag specified by the time constants of each spool. However, the magnitude of the inner-spool speed change for a change in the exhaust-nozzle area was exceedingly small so that it can be assumed that the inner spool simply does not respond to a change in the exhaust-nozzle area.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

SYMBOLS

The following symbols are used in this report:

A	exhaust-nozzle area (other areas designated by station subscript)
C	$\dfrac{d\left(\ln \dfrac{w\sqrt{T_1}}{A_1P_1}\right)}{d\left(\ln \dfrac{N_o}{\sqrt{T_1}}\right)}$
c_p	specific heat at constant pressure, 0.24
D	$\dfrac{d\left(\ln \dfrac{w\sqrt{T_2}}{A_2P_2}\right)}{d\left(\ln \dfrac{N_i}{\sqrt{T_2}}\right)}$
f	assumed functional relation between $\dfrac{w\sqrt{T_1}}{A_1P_1}$ and $\dfrac{N_o}{\sqrt{T_1}}$
G	gain term
g	assumed functional relation between $\dfrac{w\sqrt{T_2}}{A_2P_2}$ and $\dfrac{N_i}{\sqrt{T_2}}$
H	total enthalpy
I	moment of inertia
j	exponent for polytropic compression, outer compressor
K	constant corresponding to mass-flow parameter $\dfrac{w\sqrt{T}}{AP}$ under choked conditions
k	exponent for polytropic compression, inner compression
l	exponent for polytropic expansion, outer turbine
N	engine speed

P	total pressure
p	differential operator, d/dt
Q	torque
T	temperature
w	air flow
w_g	air flow plus fuel flow
γ	ratio of specific heats
η	efficiency
τ	time constant

Subscripts:

C	compressor
i	inner spool
o	outer spool
T	turbine
1	outer-compressor inlet
2	outer-compressor outlet or inner-compressor inlet
3	inner-compressor outlet
4	inner-turbine inlet
5	inner-turbine outlet or outer-turbine inlet
6	outer-turbine outlet

Transfer-function notation:

$\left.\frac{\Delta N_o}{\Delta T_4}\right|_A$ transfer function of outer-spool speed for a change in turbine-inlet temperature at constant exhaust-nozzle area

APPENDIX B

EVALUATION OF PARTIAL DERIVATIVES AT DESIGN CONDITION

Inspection of equations (5) and (6) indicates that the speed responses may be simplified considerably if some of the quantities are negligible. Evaluation of these quantities can be approximated by reference to a hypothetical two-spool engine and general thermodynamic relations between engine variables.

Assumed Engine

A two-spool turbojet engine is assumed, with the following design point characteristics:

P_2/P_1	3.5
P_3/P_2	3
P_3/P_1	10.5
w_g/w	1.012
η_T, percent	85
η_c, percent	85
Design, T_4, $^{\circ}R$	2000
Design T_4/T_1	3.85
$\gamma_{1,2,3}$	1.4
$\gamma_{4,5,6}$	1.35

Choking is assumed at both turbine inlets and at the exhaust nozzle for the design condition.

The following quantities, required for evaluation of the partial derivatives, are calculated from the assumed design conditions:

$$\frac{T_2}{T_1} = 1 + \frac{1}{\eta_C}\left[\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = 1.505 \cong \frac{H_2}{H_1}$$

$$\frac{T_3}{T_2} = 1.435 \cong \frac{H_3}{H_2}$$

$$\frac{T_3}{T_1} = 2.160 \cong \frac{H_3}{H_1}$$

$$\frac{T_4}{T_1} = 3.850 \cong \frac{H_4}{H_1}$$

$$\frac{T_3}{T_4} = 0.560 \cong \frac{H_3}{H_4}$$

At equilibrium, compressor work equals turbine work for each spool:

$$\frac{N_i Q_i}{w} = 0 = \frac{w_g}{w}(H_4 - H_5) - (H_3 - H_2) = \frac{w_g}{w} H_4 \left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2)$$

$$\frac{H_5}{H_4} = 1 - \frac{w}{w_g}\frac{H_1}{H_4}\left(\frac{H_3 - H_2}{H_1}\right) = 1 - \frac{1}{1.012} \times \frac{1}{3.85}(2.160 - 1.505)$$

$$\frac{H_5}{H_4} = 1 - 0.168 = 0.832$$

and

$$\frac{N_o Q_o}{w} = 0 = \frac{w_g}{w}(H_5 - H_6) - (H_2 - H_1) = \frac{w_g}{w}\left[\frac{H_4}{H_1}\left(\frac{H_5}{H_4} - \frac{H_6}{H_5} \times \frac{H_5}{H_4}\right)\right] - \frac{H_2 - H_1}{H_1}$$

$$\frac{H_5}{H_4}\left(1 - \frac{H_6}{H_5}\right) = \frac{H_1}{H_4}\frac{w}{w_g}\left(\frac{H_2 - H_1}{H_1}\right)$$

$$\frac{H_6}{H_5} = 1 - \frac{\frac{H_1}{H_4}\frac{w}{w_g}\left(\frac{H_2}{H_1} - 1\right)}{\frac{H_5}{H_4}} = 1 - \left(\frac{\frac{1}{3.85} \times \frac{1}{1.012} \times 0.505}{0.832}\right)$$

$$\frac{H_6}{H_5} = 1 - \frac{0.13}{0.832} = 1 - 0.156 = 0.844$$

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From H_5/H_4 and H_6/H_5, the pressure ratios P_5/P_4 and P_6/P_5 can be obtained:

$$\frac{P_5}{P_4} = \left[1 - \frac{1}{\eta_T}\left(1 - \frac{T_5}{T_4}\right)\right]^{\frac{\gamma}{\gamma-1}} = 0.433$$

$$\frac{P_6}{P_5} = \left[1 - \frac{1}{\eta_T}\left(1 - \frac{T_6}{T_5}\right)\right]^{\frac{\gamma}{\gamma-1}} = 0.455$$

$$\frac{P_4}{P_5} = 2.315$$

$$\frac{P_5}{P_6} = 2.2$$

For choking at both turbines and at the exhaust nozzle (exhaust-nozzle area constant), P_4/P_5, P_5/P_6, H_4/H_5, and H_5/H_6 are constant.

Certain functional relations are also useful in evaluation of the partial derivatives of equations (5) and (6). These are derived as follows:

$$\frac{w\sqrt{T_4}}{A_4P_4} = \text{constant} = K \qquad \text{(for choked turbine)}$$

$$\frac{w\sqrt{T_1}}{A_1P_1} = f\left(\frac{N_o}{\sqrt{T_1}}\right)$$

(The preceding relation assumes that the corrected air flow through the outer compressor is a function only of corrected speed, so that a constant corrected speed line on the compressor map would be a vertical line.)

$$\frac{\dfrac{w\sqrt{T_4}}{A_4P_4}}{\dfrac{w\sqrt{T_1}}{A_1P_1}} = \frac{K}{f\left(\dfrac{N_o}{\sqrt{T_1}}\right)}$$

If it is assumed that P_4/P_3 = constant = 0.950,

$$\frac{A_1}{A_4}\frac{P_1}{P_4}\sqrt{\frac{T_4}{T_1}} = \frac{K}{f\left(\frac{N_o}{\sqrt{T_1}}\right)} = \frac{A_1}{A_4}\frac{1}{0.950}\frac{P_1}{P_3}\sqrt{\frac{T_4}{T_1}}$$

$$\frac{A_4}{A_1}\,0.950\,\frac{P_3}{P_1}\sqrt{\frac{T_1}{T_4}} = \frac{f\left(\frac{N_o}{\sqrt{T_1}}\right)}{K}$$

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But

$$\frac{P_3}{P_1} \cong \left(\frac{T_3}{T_1}\right)^k \qquad (k = 3.075)$$

so that

$$\left(\frac{T_3}{T_1}\right)^k = \frac{f\left(\frac{N_o}{\sqrt{T_1}}\right)}{K(0.950)\,\frac{A_4}{A_1}}\sqrt{\frac{T_4}{T_1}}$$

The logarithm of both sides of the equation gives

$$k(\ln T_3 - \ln T_1) = \ln f + \frac{1}{2}(\ln T_4 - \ln T_1) - \ln\left[K(0.95)\,\frac{A_4}{A_1}\right]$$

At constant T_1 the derivative of this equation gives

$$k\,\frac{dT_3}{T_3} = \frac{df}{f} + \frac{1}{2}\,\frac{dT_4}{T_4}$$

Let $C = \dfrac{\frac{N_o}{\sqrt{T_1}}}{d\left(\frac{N_o}{\sqrt{T_1}}\right)}\dfrac{df}{f}$ and if at constant T_1, $C = \dfrac{N_o}{dN_o}\dfrac{df}{f}$

$$k\,\frac{dT_3}{T_3} = \frac{1}{2}\,\frac{dT_4}{T_4} + C\,\frac{dN_o}{N_o}$$

In a similar manner, an expression for dT_2/T_2 is obtained:

$$\frac{w\sqrt{T_2}}{A_2P_2} = g\left(\frac{N_i}{\sqrt{T_2}}\right)$$

The preceding relation assumes that the corrected air flow is a function only of the corrected speed for the inner compressor.

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$$\frac{\dfrac{w\sqrt{T_1}}{A_1P_1}}{\dfrac{w\sqrt{T_2}}{A_2P_2}} = \frac{f}{g} = \frac{A_2}{A_1}\frac{P_2}{P_1}\sqrt{\frac{T_1}{T_2}}$$

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but

$$\frac{P_2}{P_1} \cong \left(\frac{T_2}{T_1}\right)^j \qquad (j = 3.04)$$

$$\frac{f}{g} = \frac{A_2}{A_1}\left(\frac{T_2}{T_1}\right)^{j-1/2}$$

$$\ln f - \ln g = \left(j - \frac{1}{2}\right)(\ln T_2 - \ln T_1) + \ln\left(\frac{A_2}{A_1}\right)$$

$$\frac{df}{f} - \frac{dg}{g} = \left(j - \frac{1}{2}\right)\frac{dT_2}{T_2}$$

Let

$$D = \frac{N_i/\sqrt{T_2}}{d(N_i/\sqrt{T_2})}\frac{dg}{g}$$

and as previously stated,

$$C = \frac{N_o/\sqrt{T_1}}{d(N_o/\sqrt{T_1})}\frac{df}{f}$$

$$C\frac{dN_o}{N_o} - D\frac{d(N_i/\sqrt{T_2})}{N_i/\sqrt{T_2}} = \left(j - \frac{1}{2}\right)\frac{dT_2}{T_2}$$

$$C\frac{dN_o}{N_o} - \left(D\frac{dN_i}{N_i} - \frac{1}{2}D\frac{dT_2}{T_2}\right) = \left(j - \frac{1}{2}\right)\frac{dT_2}{T_2}$$

$$\left(j - \frac{1}{2} - \frac{D}{2}\right)\frac{dT_2}{T_2} = C\frac{dN_o}{N_o} - D\frac{dN_i}{N_i}$$

The terms $C = \dfrac{N_o/\sqrt{T_1}}{d(N_o/\sqrt{T_1})}\dfrac{df}{f}$ and $D = \dfrac{N_i/\sqrt{T_2}}{d(N_i/\sqrt{T_2})}\dfrac{dg}{g}$ represent the slopes of the air-flow-speed curves at constant pressure ratio (on a log-log plot) for the outer and inner compressors, respectively. Upon substitution of the values of $\frac{df}{f}$ and $\frac{dg}{g}$, the quantities C and D become 4031

$$C = \frac{d\left(\ln\dfrac{w\sqrt{T_1}}{A_1P_1}\right)}{d\left(\ln\dfrac{N_o}{\sqrt{T_1}}\right)}$$

and

$$D = \frac{d\left(\ln\dfrac{w\sqrt{T_2}}{A_2P_2}\right)}{d\left(\ln\dfrac{N_i}{\sqrt{T_2}}\right)}$$

The quantities C and D correspond to the reciprocal of the quantity B plotted in figure 3 of reference 1. From reference 1, values of $C = 1 = D$ are selected as representative in the design-speed region.

Inner-Spool Torque - Partial Derivatives

Energy available for acceleration of the inner spool is the difference between compressor and turbine work of the inner spool:

$$\frac{N_iQ_i}{w} = \frac{w_g}{w}H_4\left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2)$$

The differential form of this equation is

$$N_i \, dQ_i + Q_i \, dN_i = w \, d\left[\frac{w_g}{w} H_4 \left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2)\right] + \left[\frac{w_g}{w} H_4 \left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2)\right] dw$$

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If this change is considered to be around a steady-state operating point, $Q_i = 0$ and $\frac{w_g}{w} H_4 \left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2) = 0$ and, therefore,

$$N_i \, dQ_i = w \, d\left[\frac{w_g}{w} H_4 \left(1 - \frac{H_5}{H_4}\right) - (H_3 - H_2)\right]$$

or

$$\frac{N_i \, dQ_i}{w} = \frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right) dH_4 - dH_3 + dH_2$$

From the functional relation $Q_i = Q_i(N_i, N_o, T_4, A)$

$$dQ_i = \frac{\partial Q_i}{\partial N_i} dN_i + \frac{\partial Q_i}{\partial N_o} dN_o + \frac{\partial Q_i}{\partial T_4} dT_4 + \frac{\partial Q_i}{\partial A} dA$$

and

$$\left(\frac{N_i}{w}\right) dQ_i = \left(\frac{N_i}{w}\right)\left(\frac{\partial Q_i}{\partial N_i} dN_i + \frac{\partial Q_i}{\partial N_o} dN_o + \frac{\partial Q_i}{\partial T_4} dT_4 + \frac{\partial Q_i}{\partial A} dA\right)$$

From derivations shown previously,

$$dH_3 = \frac{H_3}{k}\left(\frac{1}{2} \frac{dT_4}{T_4} + C \frac{dN_o}{N_o}\right)$$

$$dH_2 = \frac{H_2\left(C \frac{dN_o}{N_o} - D \frac{dN_i}{N_i}\right)}{j - \frac{1}{2} - \frac{D}{2}}$$

Solution for $\partial Q_i/\partial N_i$. - At constant N_o, T_4, and A,

$$\frac{N_i}{w} dQ_i = \frac{N_i}{w} \frac{\partial Q_i}{\partial N_i} dN_i$$

and therefore

$$\frac{N_i}{w} \frac{\partial Q_i}{\partial N_i} dN_i = \frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right) dH_4 - dH_3 + dH_2$$

$$dH_3 = 0$$

$$dH_4 = 0$$

$$dH_2 = -\frac{DH_2 \dfrac{dN_i}{N_i}}{j - \dfrac{1}{2} - \dfrac{D}{2}}$$

Therefore,

$$\frac{N_i}{w} \frac{\partial Q_i}{\partial N_i} dN_i = -\frac{DH_2 \dfrac{dN_i}{N_i}}{j - \dfrac{1}{2} - \dfrac{D}{2}}$$

and

$$\frac{N_i^2}{w} \frac{\partial Q_i}{\partial N_i} = -\frac{DH_2}{j - \dfrac{1}{2} - \dfrac{D}{2}}$$

Solution for $\partial Q_i/\partial N_o$. - At constant N_i, T_4, and A,

$$\frac{N_i}{w} dQ_i = \frac{N_i}{w} \frac{\partial Q_i}{\partial N_o} dN_o$$

and

$$\frac{N_i}{w} \frac{\partial Q_i}{\partial N_o} dN_o = \frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right) dH_4 - dH_3 + dH_2$$

$$dH_4 = 0$$

$$dH_3 = \frac{CH_3}{k} \frac{dN_o}{N_o}$$

$$dH_2 = \frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}} \frac{dN_o}{N_o}$$

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Therefore,

$$\frac{N_i}{w} \frac{\partial Q_i}{\partial N_o} dN_o = - \frac{CH_3}{k} \frac{dN_o}{N_o} + \frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}} \frac{dN_o}{N_o}$$

and

$$\frac{N_o N_i}{w} \frac{\partial Q_i}{\partial N_o} = C\left(\frac{H_2}{j - \frac{1}{2} - \frac{D}{2}} - \frac{H_3}{k}\right)$$

Solution for $\partial Q_i/\partial T_4$. - At constant N_o, N_i, and A,

$$\frac{N_i}{w} dQ_i = \frac{N_i}{w} \frac{\partial Q_i}{\partial T_4} dT_4$$

and

$$\frac{N_i}{w} \frac{\partial Q_i}{\partial T_4} dT_4 = \frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right) dH_4 - dH_3 + dH_2$$

But

$$\frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right) = 1.012(1 - 0.832) = 0.170$$

$$dH_4 = c_p \, dT_4$$

$$dH_3 = \frac{1}{2} \frac{H_3}{k} \frac{dT_4}{T_4}$$

$$dH_2 = 0$$

Therefore,

$$\frac{N_i}{w}\frac{\partial Q_i}{\partial T_4}\,dT_4 = 0.17\ c_p\ dT_4 - \frac{1}{2}\frac{H_3}{k}\frac{dT_4}{T_4}$$

$$\frac{N_i}{w}\frac{\partial Q_i}{\partial T_4} = 0.17\ c_p - \frac{1}{2k}\frac{H_3}{T_4} = 0.17\ c_p - \frac{1}{2k}\,c_p\,\frac{H_3}{H_4}$$

$$\frac{N_i}{w}\frac{\partial Q_i}{\partial T_4} = c_p\left(0.17 - \frac{1}{2k}\frac{H_3}{H_4}\right)$$

Solution for $\partial Q_i/\partial A$. - At constant N_o, N_i, and T_4,

$$\frac{N_i}{w}\,dQ_i = \frac{N_i}{w}\frac{\partial Q_i}{\partial A}\,dA$$

and

$$\frac{N_i}{w}\frac{\partial Q_i}{\partial A}\,dA = \frac{w_g}{w}\left(1 - \frac{H_5}{H_4}\right)dH_4 - dH_3 + dH_2$$

$$dH_4 = 0$$

$$dH_3 = 0$$

$$dH_2 = 0$$

Therefore

$$\frac{N_i}{w}\frac{\partial Q_i}{\partial A}\,dA = 0$$

and

$$\frac{\partial Q_i}{\partial A} = 0$$

Outer-Spool Torque - Partial Derivatives

The energy available for acceleration of the outer spool is the difference between the compressor and turbine work of the outer spool:

$$\frac{N_o}{w} Q_o = \frac{w_g}{w} (H_5 - H_6) - (H_2 - H_1) = \frac{w_g}{w} H_4 \left(\frac{H_5}{H_4} - \frac{H_6}{H_5} \frac{H_5}{H_4} \right) - (H_2 - H_1)$$

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But $\dfrac{H_5}{H_4} = 0.832$

$$\frac{H_6}{H_5} = 0.844 \text{ (at constant } A)$$

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so that

$$\frac{N_o}{w} Q_o = 0.132 H_4 + H_1 - H_2$$

Linearized expansion around a steady-state operating point $(Q_o = 0)$ gives

$$\frac{N_o}{w} dQ_o = 0.132 \, dH_4 + dH_1 - dH_2$$

From the functional relation $Q_o = Q_o(N_i, N_o, T_4, A)$,

$$\left(\frac{N_o}{w}\right) dQ_o = \left(\frac{N_o}{w}\right)\left(\frac{\partial Q_o}{\partial N_i} dN_i + \frac{\partial Q_o}{\partial N_o} dN_o + \frac{\partial Q_o}{\partial T_4} dT_4 + \frac{\partial Q_o}{\partial A} dA\right)$$

Solution for $\partial Q_o/\partial N_i$. - At constant N_o, T_4, and A,

$$\frac{N_o}{w} dQ_o = \frac{N_o}{w} \frac{\partial Q_o}{\partial N_i} dN_i = 0.132 \, dH_4 + dH_1 - dH_2$$

$$dH_4 = 0$$

$$dH_1 = 0 \qquad \text{(for fixed flight conditions)}$$

$$dH_2 = \frac{- DH_2}{j - \frac{1}{2} - \frac{D}{2}} \frac{dN_i}{N_i}$$

Therefore

$$\frac{N_o}{w}\frac{\partial Q_o}{\partial N_i}\,dN_i = \frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\frac{dN_i}{N_i}$$

and

$$\frac{N_o N_i}{w}\frac{\partial Q_o}{\partial N_i} = \frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}$$

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Solution for $\partial Q_o/\partial N_o$. - At constant N_i, T_4, and A,

$$\frac{N_o}{w}\,dQ_o = \frac{N_o}{w}\frac{\partial Q_o}{\partial N_o}\,dN_o = 0.132\ dH_4 + dH_1 - dH_2$$

$$dH_4 = 0$$

$$dH_1 = 0$$

$$dH_2 = \frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}}\frac{dN_o}{N_o}$$

Therefore,

$$\frac{N_o}{w}\frac{\partial Q_o}{\partial N_o}\,dN_o = \frac{-CH_2}{j - \frac{1}{2} - \frac{D}{2}}\frac{dN_o}{N_o}$$

and

$$\frac{N_o^2}{w}\frac{\partial Q_o}{\partial N_o} = \frac{-CH_2}{j - \frac{1}{2} - \frac{D}{2}}$$

Solution for $\partial Q_o/\partial T_4$. - At constant N_o, N_i, and A,

$$\frac{N_o}{w}\,dQ_o = \frac{N_o}{w}\frac{\partial Q_o}{\partial T_4}\,dT_4 = 0.132\ dH_4 + dH_1 - dH_2$$

$$dH_1 = 0$$

$$dH_2 = 0$$

$$dH_4 = c_p\ dT_4$$

Therefore,

$$\frac{N_o}{w}\frac{\partial Q_o}{\partial T_4}\ dT_4 = 0.132\ c_p\ dT_4$$

and

$$\frac{N_o}{w}\frac{\partial Q_o}{\partial T_4} = 0.132\ c_p$$

Solution for $\partial Q_o/\partial A$. - Solution for $\partial Q_o/\partial A$ differs slightly from the preceding solutions in that H_6/H_5 is not constant for variations in exhaust-nozzle area A.

$$\frac{N_o}{w} Q_o = \frac{w_g}{w}(H_5 - H_6) - (H_2 - H_1)$$

Linearized expansion around a steady-state operating point $(Q_o = 0)$ gives

$$\frac{N_o}{w}\ dQ_o = d\left[\frac{w_g}{w}(H_5 - H_6)\right] - dH_2 + dH_1$$

The functional relation $Q_o = Q_o(N_i, N_o, T_4, A)$ gives

$$\frac{N_o}{w}\ dQ_o = \left(\frac{N_o}{w}\right)\left(\frac{\partial Q_o}{\partial N_i}\ dN_i + \frac{\partial Q_o}{\partial N_o}\ dN_o + \frac{\partial Q_o}{\partial T_4}\ dT_4 + \frac{\partial Q_o}{\partial A}\ dA\right)$$

At constant N_o, N_i, and T_4

$$\frac{N_o}{w}\ dQ_o = \frac{N_o}{w}\frac{\partial Q_o}{\partial A}\ dA = d\left[\frac{w_g}{w}(H_5 - H_6)\right] - dH_2 + dH_1$$

But

$$dH_2 = 0$$

$$dH_1 = 0$$

$$dH_5 = 0$$

Therefore,

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$$\frac{N_o}{w} \frac{\partial Q_o}{\partial A} dA = - \frac{w_g}{w} dH_6$$

With the assumption of isentropic adiabatic flow downstream of the outer-turbine,

$$\frac{w\sqrt{T_6}}{A\ P_6} = \frac{w\sqrt{T_5}}{A_5 P_5} \qquad \text{or} \qquad \frac{A}{A_5} = \frac{P_5}{P_6}\sqrt{\frac{T_6}{T_5}}$$

But

$$\frac{P_5}{P_6} \cong \left(\frac{T_5}{T_6}\right)^{l} \qquad \text{and} \quad l = 4.650$$

so that

$$\frac{A}{A_5} = \left(\frac{H_6}{H_5}\right)^{1/2-l}$$

$$\ln A - \ln A_5 = \left(\frac{1}{2} - l\right) \ln H_6 - \left(\frac{1}{2} - l\right) \ln H_5$$

$$\frac{dA}{A} = \left(\frac{1}{2} - l\right) \frac{dH_6}{H_6}$$

$$dH_6 = \frac{H_6}{A} \frac{dA}{\left(\frac{1}{2} - l\right)}$$

Therefore,

$$\frac{N_o}{w} \frac{\partial Q_o}{\partial A} = 1.012 \frac{H_6}{A} \frac{1}{\left(l - \frac{1}{2}\right)}$$

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Evaluation of Terms

From the solutions obtained for the individual partial derivatives the following terms present in equations (5) and (6) can be evaluated at design speed:

$$\frac{\frac{\partial Q_i}{\partial T_4}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial T_4}} = \frac{\left[\frac{w}{N_i} c_p \left(0.17 - \frac{1}{2k}\frac{H_3}{H_4}\right)\right]\left(\frac{w}{N_i N_o}\frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)}{\left(-\frac{w}{N_i^2}\frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)\left(\frac{w}{N_o} 0.132\ c_p\right)} = -0.6$$

(not negligible)

$$\frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}} = \frac{\left[\frac{w}{N_o N_i} C \left(\frac{H_2}{j - \frac{1}{2} - \frac{D}{2}} - \frac{H_3}{k}\right)\right]\left(\frac{w}{N_o N_i}\frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)}{\left(-\frac{w}{N_i^2}\frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)\left(-\frac{w}{N_o^2}\frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)} = 0.049$$

(negligible)

$$\frac{\frac{\partial Q_i}{\partial N_o}\frac{\partial Q_o}{\partial T_4}}{\frac{\partial Q_o}{\partial N_o}\frac{\partial Q_i}{\partial T_4}} = \frac{\left[\frac{w}{N_o N_i} C \left(\frac{H_2}{j - \frac{1}{2} - \frac{D}{2}} - \frac{H_3}{k}\right)\right]\left(\frac{w}{N_o} 0.132\ c_p\right)}{\left(-\frac{w}{N_o^2}\frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)\left[\frac{w}{N_i} c_p \left(0.17 - \frac{1}{2k}\frac{H_3}{H_4}\right)\right]} = -0.08$$

(negligible)

$$\frac{\frac{\partial Q_i}{\partial A}\frac{\partial Q_o}{\partial N_i}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial A}} = 0, \text{ because } \frac{\partial Q_i}{\partial A} = 0$$

$$\frac{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial A}}{\frac{\partial Q_i}{\partial N_i}\frac{\partial Q_o}{\partial N_o}} = \frac{\left[\frac{w}{N_o N_i} C \left(\frac{H_2}{j - \frac{1}{2} - \frac{D}{2}} - \frac{H_3}{k}\right)\right]\left[1.012\frac{w}{N_o}\frac{H_6}{A}\left(\frac{1}{l - \frac{1}{2}}\right)\right]}{\left(-\frac{w}{N_i^2}\frac{DH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)\left(-\frac{w}{N_o^2}\frac{CH_2}{j - \frac{1}{2} - \frac{D}{2}}\right)} = 0.043\frac{N_i}{A}$$

REFERENCE

1. Schmidt, Ross Dean: The Linear Dynamics of a Turbojet Engine as Developed from the Linearized Component Equations. M.S. Thesis, Univ. Minn., 1955.

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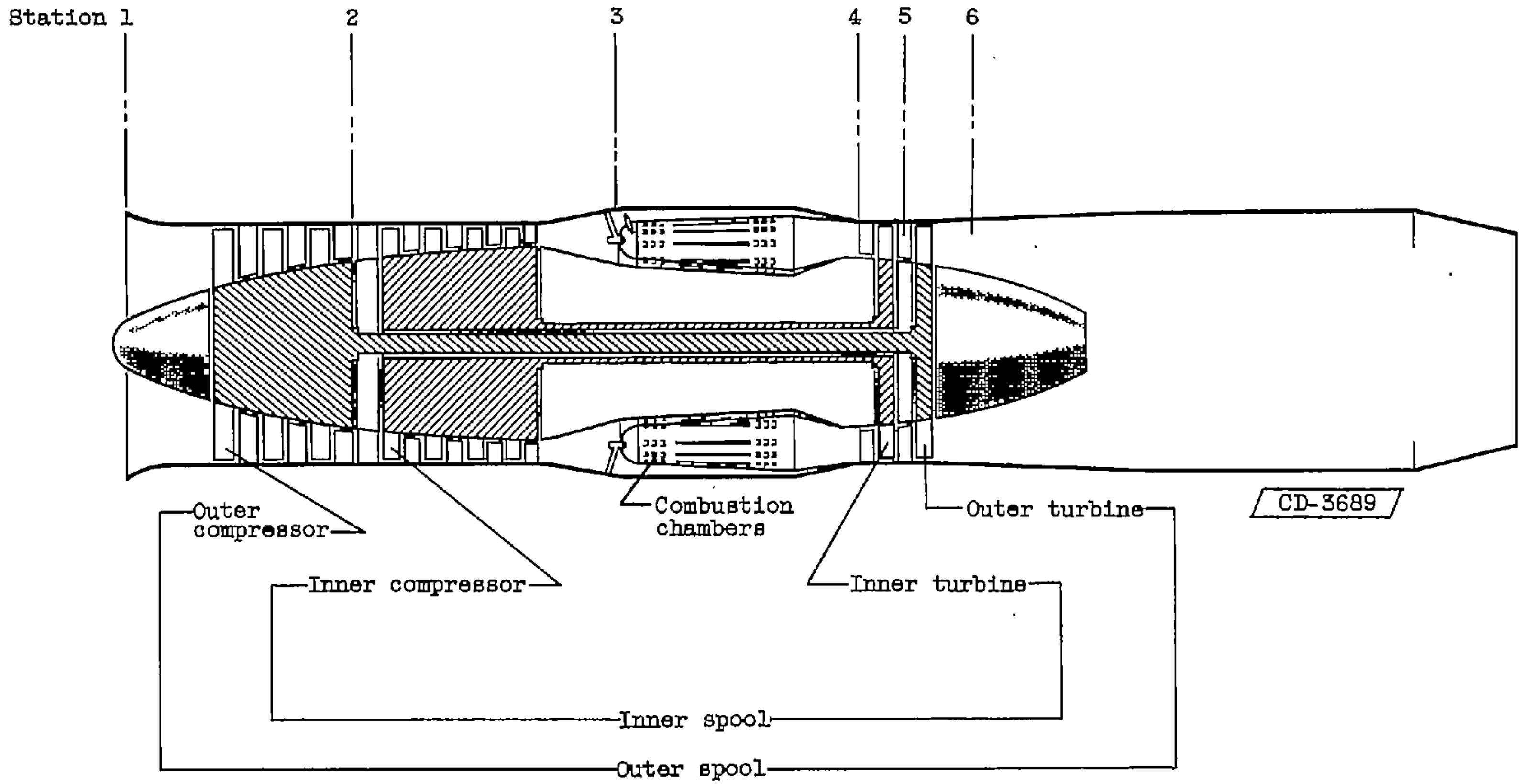

Figure 1. - Two-spool engine.

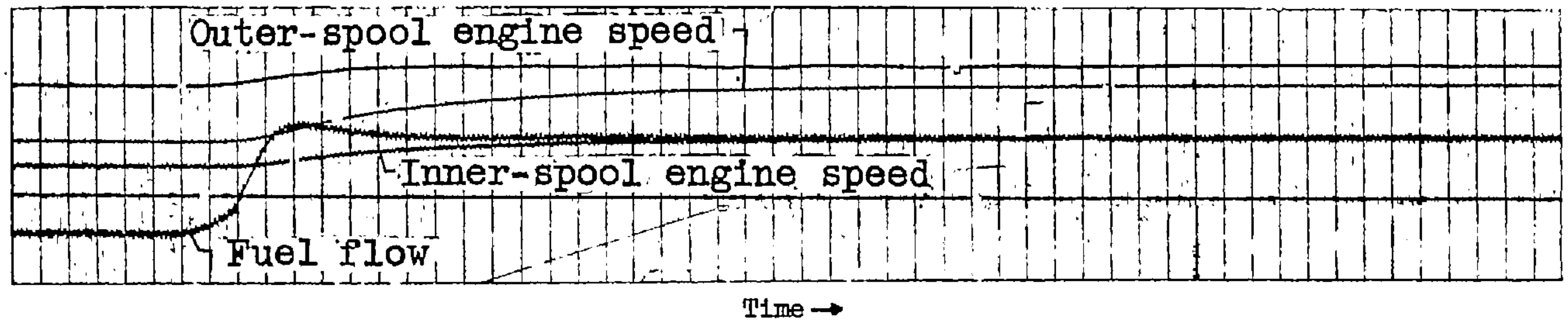

Figure 2. - Transient data for change in fuel flow at constant exhaust-nozzle area.

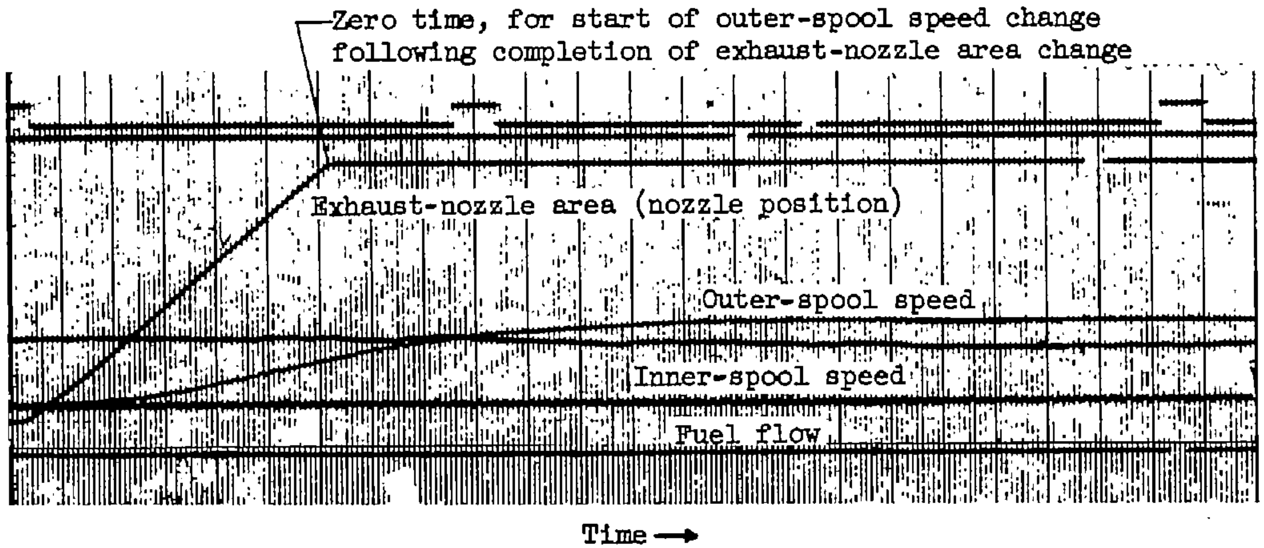

Figure 3. - Transient data for change in exhaust-nozzle area at constant fuel flow.

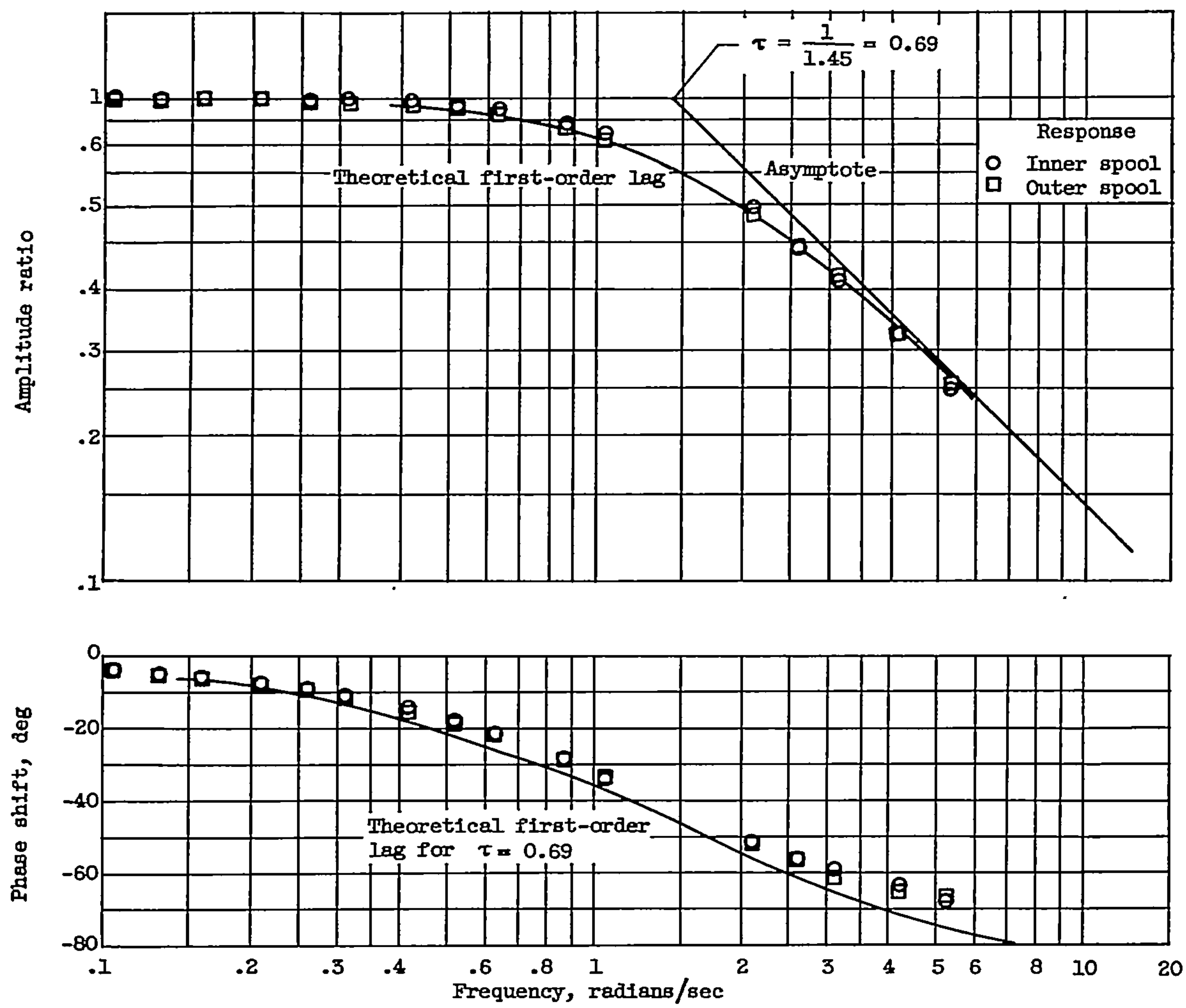

Figure 4. - Frequency response of inner- and outer-spool speed for a change in fuel plan at constant exhaust-nozzle area.

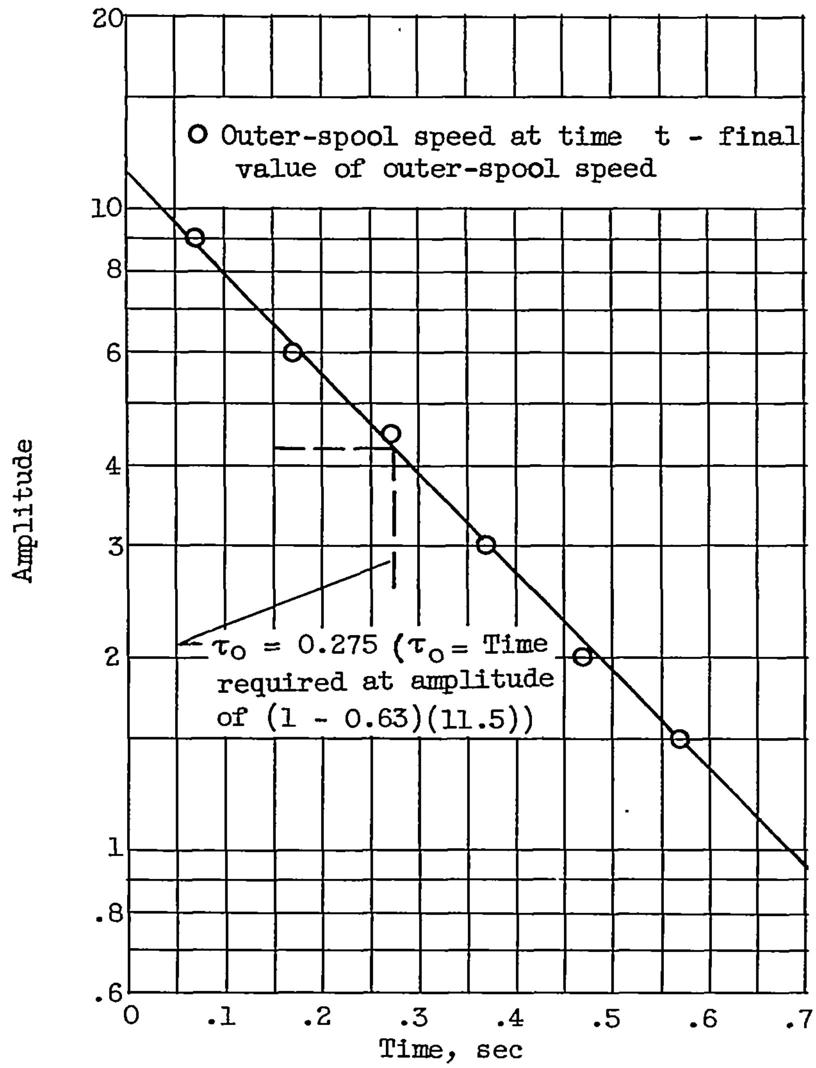

Figure 5. - Semilog plot of outer-spool speed response to exhaust-nozzle area at constant fuel flow.